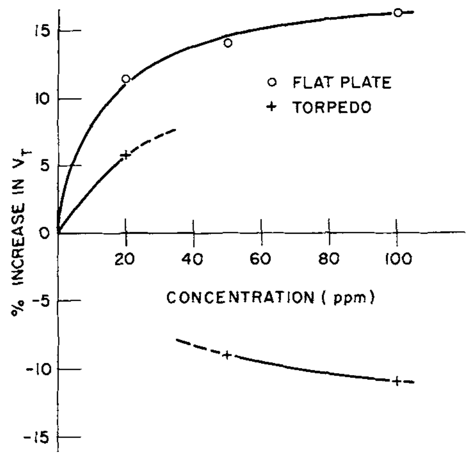

Fig. 1. Effect of additive concentration upon terminal velocity.

a lower terminal velocity! That is, the torpedo suffered a drag augmentation due to polymer addition. These data are presented in Figure 1.

Presumably the resolution of this disparity lies in the role of form drag. It seems likely that the addition of polymer so laminarized the boundary layer as to cause early separation and thus an increased form drag. In short, these experiments provide, in a modern context, a classic illustration of Eiffel's paradox (8).

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Remarks on the Withdrawal Problem for Ellis Fluids

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In a recent paper, Matsuhisa and Bird (8) have presented several solutions to non-Newtonian flow problems by using the Ellis model in the following half tau form:

$$\frac{du}{dy} = -\frac{\tau_{yx}}{\eta_o}\left[1 + \left|\frac{\tau_{yx}}{\tau_H}\right|^{\alpha-1}\right] \tag{1}$$

They discussed the withdrawal problem (5) in which a solid wall immersed in a liquid moves upward with constant velocity. Predictions of film thickness were developed as a function of rheological constants, wall velocity (u_w), and distance (z) from the free surface of the liquid. Their prediction for the unsteady state case is given by their Equation (29); in the notation of this communication, it is

$$\frac{z}{u_w t} = 1 - \frac{\rho g h^2}{\eta_o u_w}\left[1 + \left(\frac{\rho g h}{\tau_H}\right)^{\alpha-1}\right] \tag{2}$$

For the steady state case, their solution is

$$\frac{\rho g h_o^2}{\eta_o u_w}\left[1 + \left(\frac{\rho g h_o}{\tau_H}\right)^{\alpha-1}\right] = 1 \tag{3}$$

The main purpose of this communication is to note that, according to available evidence, these solutions do not properly predict film thicknesses that occur in *withdrawal*; they are, however, applicable for the less complex phenomena of *drainage*. In other words, Equations (2) and (3) are better described as solutions of the drainage problem.

A secondary purpose of this note is to compare these equations with Ellis fluid drainage equations derived elsewhere (5). A third purpose, related to the first two, is to present some minor disadvantages of the half tau form of the Ellis model.

The solutions given by Equations (2) and (3) consider only gravitational and viscous forces. However, surface tension is also important in withdrawal, as shown theoretically from 1942 on by Russian workers (6, 3, 7) and experimentally from 1946 on by workers in The Netherlands (2, 9).

The inapplicability of Equations (2) and (3) for withdrawal can be shown most easily by comparing steady state solutions for the limiting case of a Newtonian fluid. As shown in Appendix A, Equation (3) reduces to

$$h_o\left(\frac{\rho g}{\mu u_w}\right)^{1/2} = 1 \tag{4}$$

On the other hand, the best prediction of film thickness to date has been found, upon consideration of surface tension, to be

$$h_o\left(\frac{\rho g}{\mu u_w}\right)^{1/2} = 0.944\left(\frac{\mu u_w}{\sigma}\right)^{1/6}\left(1 - \frac{h_o^2 \rho g}{\mu u_w}\right)^{2/3} \tag{5}$$

Theoretical Equation (5) has been verified experimentally (*10*) over a wide range of dimensionless withdrawal speed, from a N_{Ca} of 10^{-4} to about 2. Comparison of these equations clearly indicates that the effects of speed and fluid properties on thickness predicted by Equation (4) do not represent those found in withdrawal. Furthermore, the magnitude of film thickness predicted by Equation (4) is 50 to 230% higher than that found experimentally in the capillary number range of 0.1 to 0.001; the difference is even larger at lower speeds.

The only region in which the magnitude indicated by Equation (4) seems close to reality is at higher speeds, such as $N_{Ca} = 5$. However, at the higher Reynolds numbers which occur at these speeds, waves begin to form for viscous Newtonian fluids of 50 to 600 centipoise (5). Thus, although far below the turbulent region, this condition of $N_{Ca} = 5$ is near the upper limit on the model assumption of wave-free laminar flow.

The equations neglecting surface tension may offer an upper limit on film thickness; to date at least, no experimental values have been found which exceed thicknesses predicted by Equation (4). If this is so, then the effect of surface tension on withdrawal may be explained as that of squeezing away fluid at the meniscus, thereby reducing film thickness. The reduction would be more noticeable at lower speeds, as is actually the case.

At this point, it would be natural to examine data for non-Newtonians; unfortunately, none are available in a form comparable to the half tau form of the Ellis model. Those which are available tend to confirm the above conclusion; such data are presented at the end of this note after suitable forms of the drainage equations have been presented.

It is clear that Equation (4) does not, in general, describe withdrawal for a Newtonian fluid because surface tension has been neglected; this conclusion implies that Equation (2) does not, in general, describe withdrawal for Ellis fluids for the same reason. However, drainage may be considered a special case of withdrawal when surface tension is neglected, provided that velocities are placed on a comparable basis (5). Therefore it is only necessary to compare velocities to show that Equation (2) is better described as a drainage equation than a withdrawal equation. This comparison is given below in conjunction with the secondary purpose of this note.

First, however, we cite a previously published drainage equation for Ellis fluids. It was derived from the original form of the Ellis model:

$$\frac{du}{dy} = -\tau_{yx}\,(a + b\,|\tau_{yx}|^{\alpha-1}) \qquad (6)$$

Except for the restrictions noted in Appendix *B*, the interrelationships between this coefficient form of the Ellis model and the half tau form of Equation (1) are given by the following:

$$1/a = \eta_o \qquad (7)$$

$$1/b = \eta_o\,|\tau_H|^{\alpha-1} \qquad (8)$$

Derived in terms of the coefficient or Equation (6) form of the Ellis model and based on the distance (x) from the top of the stationary plate, the drainage equation was reported (5) as

$$\frac{x}{t} = \rho g h^2\,[a + b\,(\rho g h)^{\alpha-1}] \qquad (9)$$

We now proceed to the secondary purpose of this note which is to compare drainage Equations (2) and (9). For comparison purposes, Equation (9) can be adjusted to an origin at the free surface and the withdrawal velocity by noting that the two coordinate systems are, by definition, related to time by

$$x + z = u_w t \qquad (10)$$

Substitution of Equations (7), (8), and (10) into drainage Equation (9) produces Equation (2). The steady state Equation (3) also follows from the drainage Equation (9); this can be shown either from Equation (2) or by substituting the drainage-withdrawal relationship, $u_w = x/t$, as suggested previously (5).

Applied to Ellis fluids for which all three constants are important, Equation (2) is equivalent to Equation (9). As shown in Appendix *B*, however, the half tau form of the Ellis model is not as flexible as the coefficient form in reducing to special fluids; in this sense Equation (2) is a special (less general) case of the previously reported (5) drainage equation. One example illustrating this distinction is given in the discussion following Equation (12).

That Equation (2) is very closely related to Equation (9) is not surprising. In both derivations, inertial forces (4) and surface tension were neglected in comparison to gravitational and viscous forces, the unsteady state mass balance (or continuity concept) was invoked, and the boundary condition in the vertical direction was taken as zero film thickness at the top. Thus, although the derivations for the two equations were presented in different order, notation, and coordinate system, all the assumptions and methods used were equivalent or interrelated in a simple way.

With the Ellis model drainage Equation (9) available in the coefficient form, it is now possible to place in comparable form both the drainage prediction and the available withdrawal data for a power law fluid ($a = o$, $1/b = k^\alpha$). Thus we return to information concerning the main purpose of this note. The predicted film thickness using drainage Equation (9) is given (5) as

$$h_o = \left(\frac{K^\alpha u_w}{\sigma(\rho g)^\alpha}\right)^{\frac{1}{\alpha+1}} \qquad (11)$$

The available Carbopol withdrawal data have been correlated by using theoretically predicted functions and empirically determined constants. The correlation (5) may be expressed as

$$h_o = \left(0.41 + \frac{0.25}{\alpha}\right)\left[\frac{K^{4\alpha}u_w^{4}}{\sigma^\alpha(\rho g)^{3\alpha}}\right]^{\frac{1}{4+2\alpha}} \qquad (12)$$

where $1 \leqq \alpha \leqq 3$

Comparison of Equations (11) and (12) indicates that the drainage equation is, in general, inadequate for predicting the effect of fluid properties and speed in withdrawal for one type of non-Newtonian model. Furthermore, as shown graphically (5), the disagreement in magnitude of the thickness is larger for these fluids than for Newtonian fluids. Although Equation (2) cannot be tested directly with these data, since it does not reduce to a power law fluid (see Appendix *B*), these non-Newtonian results tend to substantiate the Newtonian fluid conclusions reported above.

NOTATION

- a,b = rheological constants; coefficient form of the Ellis model, Equation (6)
- g = acceleration of gravity
- h = film thickness at any point
- h_o = film thickness at steady state
- k = rheological constant, power law, $k^\alpha = 1/b$
- N_{Ca} = capillary number, $\mu u_w/\sigma$
- t = time

u = fluid velocity at any point
u_w = withdrawal velocity
x = distance down from top of the plate
y = distance perpendicular to the solid
z = distance up from liquid surface

Greek Letters

α = rheological constant
η = local viscosity, non-Newtonian
η_o = rheological constant, local viscosity at zero shear, Equation (1)
μ = viscosity of Newtonian fluid
ρ = density
σ = surface tension
τ_{yx} = shear stress
τ_H = rheological constant, shear stress which occurs when $\eta = 0.5\ \eta_o$; half tau form of the Ellis model, Equation (1)

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APPENDIX A

The Equation (1) form of an Ellis fluid reduces to a Newtonian fluid when $\tau_H \to \infty$. Since $\eta_o = \mu$ for this case, Equation (3) becomes

$$h_o^2\left(\frac{\rho g}{\mu\, u_w}\right) = \lim_{\tau_H \to \infty} \frac{1}{\left[1 + \left(\frac{\rho g h_o}{\tau_H}\right)^{\alpha - 1}\right]} \qquad (A1)$$

But

$$\lim_{\tau_H \to \infty} \left(\frac{\rho g h_o}{\tau_H}\right)^{\alpha - 1} = 0 \qquad (A2)$$

Equation (*A*2) indicates that the right side of Equation (A1) becomes unity. Thus Equation (*A*1) simplifies to Equation (4), the result obtained by Jeffries and others, as reported elsewhere (*4*).

The equation given by Matsuhisa and Bird (*8*) for the Newtonian case differs from Equation (4) by a numerical constant. Their error was apparently due to taking the limit in Equation (*A*2) as $\alpha \to 1$, while implicitly assuming that τ_H remained finite. Such an error leads to a value of ½ for the right side of Equation (A1).

Upon examining the definition of the constants for this half tau or Equation (1) form of the Ellis model, it might seem that the proper condition for a Newtonian fluid would be $\tau_H \to \infty$ and $\alpha \to 1$ simultaneously. Such a limiting process applied to Equation (2) leads to the indeterminant form (0°); proper evaluation could then be made by noting that $\tau_H \to \infty$ at a faster rate than $\alpha \to 1$. Upon further reflection, however, the original simple statement that $\tau_H \to \infty$ is preferred (with at finite α added if desired); this leads to the correct solution in a straightforward way.

Parenthetically, it should be noted that the complete statement by Matsuhisa and Bird regarding the Newtonian case (*8*) is accurate. However, this is due to an unexplained and apparently improper switch from a Newtonian viscosity of η_o to $\eta_{o/2}$, combined with the α limitation error noted above.

In order to avoid the α limitation error, special care must be exercised in finding the Newtonian case when using this half tau form or Ellis model. On the other hand, the coefficient of the Equation (6) form of the Ellis model reduces to the Newtonian case by use of vanishing coefficients (5); possible evaluation errors noted in the half tau form are avoided. Therefore, for any geometry, the half tau form has the minor disadvantage of requiring special care to avoid numerical errors. This requirement is apparently due to implied restrictions in the definitions of τ_H, as discussed in Appendix *B*.

APPENDIX B

The Ellis model fluid has three rheological constants: the exponent and two others. Equation (6) presents the model in original coefficient form of two arbitrary constants (a, b). For reasons given elsewhere, the Equation (1) or half tau form is based on constants (η_o, τ_H) defined in physical terms.

The use of these physical terms places no restriction on the Ellis model as long as all three parameters are important (neglecting, for this discussion, the wider range of data that may be needed to obtain meaningful values of η_o and τ_H). However, as compared with the coefficient form, the half tau form does have less flexibility in simplifying to special cases of 1, 2, and 3 constant fluids. Some specific examples are:

1. The half tau form does not reduce to the equations for a Bingham plastic. This is due to the definition of τ_H, as τ_H must always be positive to be consistent with the physical meaning ascribed to it. This difference between Equations (1) and (6) was apparently first noted by Bird (*1*). On first glance, it might appear that this difference could be avoided by relaxing the physical meaning on τ_H. In so doing, however, one in effect returns to the coefficient form, thereby indicating that the half tau form is less flexible for this case.

2. The half tau form does not reduce to a power law fluid. This is due to the appearance of η_o in the b coefficient of Equation (8), as well as the a coefficient of Equation (7). For any specified geometrical application, this restriction prevents a direct comparison of power law data with a half tau form model; for example, this restriction prevents a direct comparison of Equation (12) with Equation (2). Of course one could make such a comparison by returning to the coefficient form or its equivalent; this amounts to relaxing the physical meaning and to discarding the half tau form, however. Other ways of obtaining a comparison of such data are possible but do not involve the power law model.

There is a temptation to say that Equation (1) reduces to a power law fluid where $\tau_H \to 0$; at finite η_o, however, this leads to the unreasonable conclusion that the motion of all power law fluids occurs at an infinite velocity gradient. An alternative approach based on $\eta_0 \to \infty$ (at finite τ_H) leads to a similarly unreasonable conclusion of zero velocity gradients for all power law fluid motion. However, the equivalent of these two approaches may be used if one relaxes physical meaning and returns to the form of Equation (6).

A possible criticism of this restriction might be that power law fluids are, strictly speaking, nonexistent. However true this is, it does not change the first sentence in this second example.

3. Care must be used to reduce the half tau form to a Newtonian fluid or a numerical error may result (see Appendix *A*). This is due to the placement of τ_H in the denominator and dependence of τ_H on η_o and $\eta(\tau_{yx})$ for purposes of evaluation of τ_H from data.

The coefficient form of the Ellis model is not restricted by the physical meanings implied by Equations (7) and (8). As a result, the coefficient form has more flexibility in reducing to the special cases noted in examples 1 to 3. In this sense the half tau form is less general. Furthermore, Equation (1) may be considered a special case of Equation (6) but the reverse is not true. For some conditions, however, Equations (1) and (6) may be considered equivalent.

To view this Appendix *B* in proper perspective, one should realize that there are valid reasons and benefits in using the half tau form. These have been discussed in detail (*8*). Since this form seems to be used more frequently, however, it is noteworthy that it has a few minor restrictions or disadvantages. It is equally important to note that in the long run the advantages of the half tau form probably outweigh these disadvantages.